Incentives for Efficient Price Formation in Markets with Non-Convexities

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Errata:

On Sept. 25, 2019, the paper was amended and reposted to make the following corrections:

1. Equation (56) was changed to be conditioned on (47).
2. Equations (56) – (57) were changed to be conditioned on (47) – (48).
3. On the line before Equation (66), the variable w_i was changed to z_i.
4. In Table 1, the maximum generation for the unit at Node 2 was changed to 50 MW (instead of 100 MW).

Contents

Abstract

This paper examines the incentives for efficient pricing mechanisms in markets with non-convexities. The wholesale electricity market is a prominent case in which non-convexity has emerged as a critical issue. Ideally, an efficient pricing mechanism produces market signals that reflect costs and scarcities, incents price-taking behavior and yields sufficient revenues to attract new investment. However, under non-convex conditions, there is no assurance that these goals can be fully achieved, and market equilibrium may not even exist. Previous studies on markets with convexities have been focusing on the revenue-sufficiency problem. Positive results on incentives are relatively scarce. This paper intends to fill the gap. With non-convexities, quasi-equilibrium entails solving separately a non-convex allocation model and a convexified pricing model with solution-support payments in settlement. We consider three convex-relaxation methods, including Lagrangian dualization, convex-hull relaxation and integer relaxation.[1] We show that quasi-equilibrium pricing is dominant strategy incentive compatible (DSIC) in the limit and the total side payment, divided by the total surplus, approaches zero when the market size (measured by the number of consumers) increases to infinity. In essence, the quasi-equilibrium pricing mechanism extends efficient pricing principles from a convex market environment to one that is non-convex in ways that preserve economic efficiency, incentive compatibility and revenue sufficiency.

These results are illustrated in the context of wholesale electricity markets. Since 2014, price formation issues have been vigorously debated in the U.S., including FERC's conferences and proceedings with comments from academics, policy and business communities across ISO/RTO regions. Convex-hull pricing is generally considered an ideal solution, but it remains computationally prohibitive. In this paper, we identify conditions under which the integer relaxation method can produce close and sometimes even exact approximations to convex-hull pricing. In April 2019, FERC authorized the use of integer relaxation as a just and reasonable pricing method for fast-start units in PJM's energy markets.

[1] Integer relaxation refers to a convex relaxation of an MIP problem in which the integer variables are linearized.

1. Introduction

In this paper, we examine the incentive properties of efficient pricing mechanisms in markets under non-convex conditions. The wholesale electricity market is a prominent case. During the past two decades, the path-breaking electricity market reform in the U.S. has generally been considered a successful experience. Nonetheless, complex socio-economic and technological realities have evolved in ways that put the market design under an increasingly more rigorous test. Non-convexity has emerged as a critical issue given the trends of lowering fuel costs, declining demand growth and the rapid growth of renewables and environmental externalities of fossil fuels, testing the robustness and effectiveness of the existing pricing mechanism. Electricity markets present a unique challenge, due to the necessity that demand and supply must be in balance every minute on a vast transmission network in order to keep the lights on. Rapid changes in technology and industry structure would lead to the continued evolution of market design.

Path-Breaking Theories

The foundation for electricity market design is built on path-breaking developments in economics, engineering and operations research since the 1950s. The general equilibrium theory provides a basic framework for how the price system forms market signals in support of efficient allocation. Fundamental theorems of welfare economics set out conditions under which competitive markets produce efficient allocation. One of the necessary conditions is known as convex preferences and production sets for consumers and producers.[2] Under convex conditions, the market price would equate marginal cost and marginal value at competitive equilibrium, ensuring that each consumer's surplus and each producer's surplus is maximized, and the social welfare is at a maximum.[3] In essence, the pricing mechanism plays the role of the Arrovian "invisible hand."[4] Ideally, an efficient pricing mechanism would achieve three desirable outcomes: (1) efficient allocation – market price signals reflect costs and scarcities in ways that achieve the maximum social welfare; (2) incentive compatibility – the optimal strategy for each agent is to behave truthfully and act as a price taker with no incentive to deviate from efficient allocation; and (3) revenue sufficiency – market revenue provides appropriate compensation to cover the investment costs of new entry. However, under non-convex conditions, it is theoretically impossible to achieve these results simultaneously.

With the introduction of the simplex method, linear programming has played a central role in convex optimization.[5] The duality theory indicates that market allocations and efficient prices could be jointly determined from the primal and dual solutions for a convex optimization problem. Later, with the rapid growth of game theory and its applications to competitive auctions, incentive has been recognized as an essential feature for market design. Under non-convexity conditions, incentive compatibility warrants especially close attention. In particular, the innovative Vickrey auction design, with its signature separation of the selection (first-best) and the pricing (second-best) rules in auction design, has laid the

[2] Under convex conditions, the average cost (as well as the incremental cost) of production does not decline when the output increases and does not rise when the output decreases.

[3] Arrow and Debreu, 1954; Arrow and Hahn (1971)

[4] As is well known, Adam Smith created the famous metaphor of the "invisible hand" with the conjecture that the invisible hand will guide the efficient exchange of scarce resources through competition in the marketplace. The late Stanford University Professor Kenneth J. Arrow, a winner of the Economics Nobel Prize in 1972, offered fundamental insights in social choice theory laying the foundation for fundamental theorems of welfare economics setting out the precise conditions under which the "invisible hand" conjecture holds true, so that the general competitive equilibrium can achieve efficient outcome and maximize social welfare.

[5] Dantzig (1948)

theoretical foundation of market mechanism design.[6] The Vickrey-Clark-Groves (VCG) mechanism sets a high standard with the notion of dominant strategy incentive compatibility.

In the 1980s, the development of the homeostatic control theory laid the theoretical foundation for integrating marginal cost pricing principles into system control of the electric grid, based on locational marginal pricing (LMP).[7] In the 1990s, the landmark Federal Energy Regulatory Commission (FERC) Orders 888 and 889 on electricity market restructuring adopted the LMP framework, combined with financial transmission rights in the "standard" market design. In 1997, PJM developed market rules that embody these market principles (Ott 2003; PJM 2017). At the time, for practicality, the current LMP-based pricing mechanism was adopted as a simple and reasonable approximation for efficient pricing.[8] For over twenty years, it has been accepted by regulators as a just and reasonable pricing method. Also at the time of electricity restructuring, many important market design issues called for attention in public policy and academic forums, including the bidding format. In the presence of non-convex cost structure with variable energy cost and fixed-commitment costs, the three-part bid format includes energy, no-load and start-up costs,[9] while a one-part bid format was generally considered simpler and more compatible with the marginal-cost pricing principle. However, a three-part bid format provides useful information to the system operator for making efficient allocation decisions for economic commitment and dispatch in pool-based markets. However, a three-part bid structure creates non-convexity issues that complicate the design of the pricing mechanism.

Non-Convexities in Electricity Markets

In a convex market environment, a system operator would dispatch generation smoothly, starting with the cheapest resource and then include more expensive resources until demand is fully met (after considering constraints imposed by the transmission system and generator constraints such as minimum run times), with all generation paid at the price of the marginal cost of the most expensive resource dispatched. Without market power, generators do not have much to gain by bidding manipulatively, and the system operator would be able to dispatch at the lowest possible cost. Each price-taking unit would receive the greatest possible profit by bidding in a way that reflects its true costs and operating constraints, while consumers would benefit.

Without non-convexities, the dispatch and pricing solutions could simply be obtained from the primal and dual solutions of the same security-constrained unit commitment and economic dispatch (SCUC/SCED) model run. There would be no uplift, no "missing money," no revenue insufficiency, no incentive for self-scheduling, no reward for inflexibility, and no fall in prices when demand increases. In reality, evidence across the organized wholesale electricity markets indicates that the effects of non-convexities are prevalent. For example, at PJM, nearly one-fifth of the time, the price falls when demand increases, a paradox that does not exist in a well-founded market. The price suppression effects due to non-convexity would result in revenue insufficiency.

In electricity markets, non-convexity arises for other technical reasons, such as fixed start-up/no-load costs, economies of scale and inflexibilities such as minimum-generation or block-loading requirements. Under non-convex conditions, units that are economically selected to serve load may incur losses if the price is set at marginal cost. With non-convexities,

[6] Vickrey (1962)

[7] Schweppee et al. (1980), Hogan (1992)

[8] See O'Neill., Helman, Sotkiewicz, Rothkopf, and Stewart (2001)

[9] See Chao and Huntington (1998).

there have always been circumstances where the prices could not reflect everything relevant to sending the right market signals. The potential for such problems has been known since the beginning of the wholesale electricity markets. For example, it has been observed that a significant number of combustion turbines scheduled their daily bids in unit parameters, which are less flexible than the original equipment manufacturer data, while units are not rewarded for offering flexibility to the market. Such incentives may exacerbate market-power behavior in the presence of local transmission constraints. Some units have an incentive to offer in a manner that maximizes a potential uplift payment (for example, by claiming a longer minimum run time). Although such bidding behavior may be rational from the perspective of each individual supplier, collectively, it could cause distorted price formation.[10]

Pricing under Non-Convexities

Non-convexity presents a fundamental challenge for pricing mechanisms to ensure efficient allocation in competitive markets. As Wilson (1993) has aptly observed, non-linear pricing is an integral part of efficient pricing mechanisms in market environments with non-convexities and other market imperfections.[11] Previous studies on pricing mechanisms in electricity markets have largely been focused on ways to minimize the use of side payments, also known as uplift payments. Under non-convex conditions, LMPs alone would not be sufficient to support the dispatch solution. Some units that are needed to serve load may incur losses, while other units that are not selected may be profitable to self-schedule. Ring (1995), later joined by Gribik, Hogan and Pope (2007), began work on extending the basic LMP to minimize uplift payments that create incentives for inefficient manipulative behavior. Their collective work, along with other scholarly contributions, laid the foundation for an approach known as extended locational marginal pricing (ELMP),[12] which was implemented in a limited form by the New York Independent System Operator (NYISO) in 1998 and the Midcontinent Independent System Operator (MISO) in 2015. The ELMP, or convex-hull pricing (Gribik et al. 2007), provides a way to determine electricity market clearing prices that minimizes the total uplift payment. However, convex-hull pricing presents computational requirements that would be challenging under the best of circumstances, and it would be even more challenging to apply in the short time frame required for the real-time market. O'Neill et al. (2005) presented alternative approaches, based on a general market formulation based on mixed-integer linear programming.

While previous studies on pricing mechanisms under non-convex conditions have largely focused on minimizing the use of side payments, results on incentive compatibility have been relatively scarce. To fill the gap in the literature, this paper focuses on efficient pricing mechanisms based on the notion of quasi-equilibrium in the seminal work of Starr (1969) and Arrow and Hahn (1971). Quasi-equilibrium addresses the issue that the general market equilibrium may not exist in the presence of non-convex preferences in two steps. First, a revised general equilibrium problem is formed through replacing the non-convex preferences by their convexified approximations to remove the non-convexities, a procedure known as convex relaxation. Then, the revised economy is solved for market equilibrium so that at the quasi-equilibrium prices, all consumers can choose an optimal set of consumer goods within their budget constraints, and the market for every good clears in the revised market. By invoking the Shapley-Folkman theorem, Starr (1969) shows that in a quasi-equilibrium, the prices "nearly clear" the markets for the original economy in the sense that the "distance" between the market allocation of

[10] Chao, H. (2018) Challenges for Getting the Prices Right in PJM's Wholesale Electricity Markets, Harvard Energy Policy Seminar, March 26. Retrieved from https://sites.hks.harvard.edu/m-rcbg/cepr/HKS%20Energy%20Policy%20Seminar%20-%20Chao%2020180326.pdf . Chao, H. (2019) Electricity market reform to enhance the energy and reserve pricing mechanism: Observations from PJM, Energy Systems Workshop at Isaac Newton Institute, University of Cambridge, January 7, 2019. Retrieved from http://www.newton.ac.uk/files/seminar/20190107160017001-1481148.pdf

[11] Unlike linear pricing which maintains the same price per unit, nonlinear pricing includes multi-part pricing, price menu with differentiated options, and a variety of price schedules with nonlinear structures.

[12] See Ring (1995), Hogan and Ring (2003); Sioshansi, R., R. O'Neill, and S. Oren (2008); and Gribik, Hogan, and Pope (2007).

the convexified economy and that of the original non-convex economy is small, on average, as the number of consumers goes to infinity. As one of the fundamental insights on economic incentives, Hurwicz (1972) shows that with a finite number of agents (consumers) in an exchange economy, no market mechanism based on uniform pricing can assure incentives for competitive price-taking behavior. Postlewaite and Roberts (1976) show that in large economies, incentive compatibility for price-taking behavior would prevail when the number of consumers increases to infinity.

This paper contributes two new insights. First, in the presence of non-convexities, quasi-equilibrium pricing comprises separated allocation and pricing models with solution-support settlements. We show that quasi-equilibrium pricing converges to a VCG mechanism and is DSIC in the limit, while the total side payment divided by the social surplus approaches zero when the market size (measured by the number of consumers) increases to infinity. Second, while the integer relaxation method is simple for practical implementation, we identify conditions under which it can produce good, or even exact, approximations to convex-hull pricing. The new insights are extended to pool-based wholesale electricity markets, in which an economic commitment and dispatch model, or simply the dispatch model, is used to determine an optimal generation schedule and dispatch of generating units, and the pricing model – an integer relaxation of the dispatch model – is used to determine the compensation of dispatched generating units. In April 2019, FERC authorized the use of integer relaxation as a just and reasonable pricing method for fast-start units in PJM's energy markets.[13] In essence, a quasi-equilibrium pricing mechanism extends efficient pricing principles from a convex to a non-convex market environment in ways that ensure economic efficiency, incentive compatibility and revenue sufficiency.

The remaining sections of the paper are organized as follows. Section 2 presents the basic market framework, including the allocation model, the pricing model and the solution support settlement. Section 3 presents the results on incentive compatibility. Section 4 presents the electricity market framework, including the dispatch model, the pricing model and the settlement rule. Section 5 discusses implications in practice and topics for future research. Section 6 concludes with a brief summary.

2. Basic Market Framework

In this section, we present the basic framework of market pricing mechanisms in markets with non-convexities, based on the concept of quasi-equilibrium. The market model comprises three components: the allocation model, the pricing model and the settlement rule. The allocation model is formulated as a mixed integer programming (MIP) problem with the objective of maximizing the social welfare, measured by the sum of consumer's surplus and producer's surplus. The pricing model is a convex relaxation of the market allocation model. We consider three different convex relaxation approaches, Lagrangian dualization, convex-hull relaxation and integer relaxation, which differ in computational difficulty. The settlement rule entails the provision of side payments to implement the allocation and pricing solutions.

Assumptions

Consider a market consisting of I consumers (indexed by $i = 1, \dots, I$), J producers (indexed by $j = 1, \dots, J$) and K commodities (indexed by $k = 1, \dots, K$). Let $\boldsymbol{p} \equiv (p_1, \dots, p_K) \in R^K$ be the price vector, $\boldsymbol{x}_i \equiv (x_{i1}, \dots, x_{iK}) \in R^K$ be

[13] PJM (2019a, 2019b)

the vector of the consumption levels for each consumer, $\boldsymbol{y}_j \equiv (y_{j1}, \dots, y_{jK}) \in R^K$ be the vector of the production levels for each producer, and $z_j \in Z \equiv \{0,1\}$ denote a binary integer variable associate with a fixed cost.

We assume that each consumer's preference over consumption is represented by a quasi-linear utility function $u_i(\boldsymbol{x}_i)$. Given the assumption of quasi-linear preferences, each consumer's demand function does not have wealth effects. We further assume that for each consumer $u_i(\cdot)$ is a differentiable, non-decreasing concave function. Given the price vector $\boldsymbol{p}$, the consumer's surplus is written:

$$\varphi_i(\boldsymbol{x}_i, \boldsymbol{p}) \equiv u_i(\boldsymbol{x}_i) - \boldsymbol{p} \cdot \boldsymbol{x}_i \tag{1}$$

The maximum consumer's surplus, or the indirect utility function, equals the negative of the conjugate of a concave utility function:

$$\hat{\varphi}_i(\boldsymbol{p}) \equiv \underset{\boldsymbol{x}_i \in R^K}{Max}\, \varphi_i(\boldsymbol{x}_i, \boldsymbol{p}) = -\underset{\boldsymbol{x}_i \in R^K}{Min}\{\boldsymbol{p} \cdot \boldsymbol{x}_i - u_i(\boldsymbol{x}_i)\} \tag{2}$$

Note that $\hat{\varphi}_i(\cdot)$ is a convex function**.**

Let Θ denote the set of technology types. We assume that Θ is a compact set, and $\theta_j \in \Theta$ denotes a producer's technology type. The producer's cost function is denoted by $c(\boldsymbol{y}_j, z_j|\theta_j)$ or simply $c_j(\boldsymbol{y}_j, z_j)$. We assume that for any given z_i, $c_j(\cdot, z_j)$ is a differentiable convex function. The convex envelope of the cost function is denoted by $\check{c}_j(\boldsymbol{y}_j, z_j)$. Given the price vector $\boldsymbol{p}$, the producer's surplus or profit is written:

$$\pi_j(\boldsymbol{y}_j, z_j, \boldsymbol{p}) \equiv \boldsymbol{p} \cdot \boldsymbol{y}_j - c_j(\boldsymbol{y}_j, z_j) \tag{3}$$

The following maximum producer's surplus functions are conjugates of the cost function:

$$\pi_j^*(z_j, \boldsymbol{p}) \equiv \max_{\boldsymbol{y}_j \in R^K} \pi_j(\boldsymbol{y}_j, z_j, \boldsymbol{p}) \tag{4}$$

$$\hat{\pi}_j^{LD}(\boldsymbol{p}) \equiv \max_{\boldsymbol{y}_j \in R^K, z_j \in Z} \pi_j(\boldsymbol{y}_j, z_j, \boldsymbol{p}) \tag{5}$$

$$\hat{\pi}_j^{CH}(\boldsymbol{p}) \equiv \max_{\boldsymbol{y}_j \in R^K, z_j \in \bar{Z}} \boldsymbol{p} \cdot \boldsymbol{y}_j - \check{c}_j(\boldsymbol{y}_j, z_j) \tag{6}$$

$$\hat{\pi}_j^{IR}(\boldsymbol{p}) \equiv \underset{\boldsymbol{y}_j \in R^K, z_j \in \bar{Z}}{max} \pi_j(\boldsymbol{y}_j, z_j, \boldsymbol{p}) \tag{7}$$

As conjugate functions, the maximum producer's surplus functions in (4) – (7) are convex functions in $\boldsymbol{p}$. Note that in (5), z_j is an integer variable, though in (6) and (7), z_j is relaxed to become a real variable. In (6), the original cost function is replaced by its convex envelope.

The integer variable z_j can be interpreted as the entry decision for firm *j*. We assume that the cost function for each producer $c_j(\boldsymbol{y}_j, z_j)$ is positive homogeneous of degree one in $(\boldsymbol{y}_j, z_j)$. Next, we show below that the conjugates of the cost function are equivalent under the three convex relaxation methods.

Lemma 1. If the cost function $c_j(\boldsymbol{y}_j, z_j)$ is positive homogeneous of degree one in $(\boldsymbol{y}_j, z_j)$, then $\hat{\pi}_j^{IR}(\boldsymbol{p}) = \hat{\pi}_j^{LD}(\boldsymbol{p}) = \hat{\pi}_j^{CH}(\boldsymbol{p}) \equiv \hat{\pi}_j\ (\boldsymbol{p})$.

Proof: First, we define, for each firm *j*,

$$\boldsymbol{y}^*(z_j, \boldsymbol{p}) \in \underset{\boldsymbol{y}_j \in R^K}{\operatorname{argmax}} \pi_j(\boldsymbol{y}_j, z_j, \boldsymbol{p}) \tag{8}$$

Then, for any $\alpha > 0$, we have:

$$\begin{aligned} \boldsymbol{y}_{\boldsymbol{j}}^*(\alpha z_{\boldsymbol{j}}, p) &\in \underset{\boldsymbol{y}_j \in R^K}{\operatorname{argmax}} \pi_j(\boldsymbol{y}_j, \alpha z_{\boldsymbol{j}}, \boldsymbol{p}) = \underset{\alpha \boldsymbol{y}_j \in R^K}{\operatorname{argmax}} \pi_i(\alpha \boldsymbol{y}_j, \alpha z_j, \boldsymbol{p}) \\ &= \underset{\alpha \boldsymbol{y}_j \in R^K}{\operatorname{argmax}} \alpha \pi_j(\boldsymbol{y}_j, z_j, \boldsymbol{p}) \end{aligned} \tag{9}$$

By substituting the above result in (3) - (4), we obtain:

$$\pi_j^*(\alpha z_j, \boldsymbol{p}) = \pi_j(\boldsymbol{y}_j^*(\alpha z_j, \boldsymbol{p}), \alpha z_j, \boldsymbol{p}) = \alpha \pi_j(\boldsymbol{y}_j^*(z_j, \boldsymbol{p}), z_j, \boldsymbol{p}) = \alpha \pi_j^*(z_j, \boldsymbol{p}). \tag{10}$$

Equation (9) implies that $\pi_j^*(z_j, \boldsymbol{p})$ is linear in z_j. Since the relaxed model is a linear program, the result follows from the fundamental theorem of linear programming: If feasible, it should always be possible to find an optimal integer solution at one of the extreme points in the constraint set of a unit interval, $\bar{Z} = [0,1]$. QED

Given the assumption of quasi-linear utility functions, there are no wealth effects. Without loss of generality, the general equilibrium problem for competitive markets can be studied within the simpler partial equilibrium framework. The competitive equilibrium is defined as a price vector, a consumption vector for each consumer that maximizes the consumer's surplus at those prices, and a production vector for each supplier that maximizes the producer's surplus at those prices, and the market-balance condition under which the demand equals the supply. This is stated formally as follows:

Definition: The allocation vector $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$ and price vector $\boldsymbol{p}^*$**,** constitute a competitive equilibrium if the following conditions are satisfied:

a) Utility maximization: Each consumer chooses $\boldsymbol{x}_{\boldsymbol{i}}^*$ that maximizes the net utility, or the consumer's surplus, $\varphi_i(\boldsymbol{x}_i, \boldsymbol{p}^*)$.[14]

$$\boldsymbol{x}_{\boldsymbol{i}}^* \equiv \underset{\boldsymbol{x}_i \in R^K}{argmax}\, u_i(\boldsymbol{x}_i) - \boldsymbol{p}^* \cdot \boldsymbol{x}_i = \underset{\boldsymbol{x}_i \in R^K}{argmax}\, \varphi_i(\boldsymbol{x}_i, \boldsymbol{p}^*) \tag{11}$$

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[14] Under the assumption of a quasi-linear utility function with a numeraire good denoted by x_i^0 and maximum expenditure by m_i, a consumer's problem of utility maximization subject to budget constraint is equivalent to the maximization of the consumer's surplus absent the budget constraint:

$$\underset{x_i^0 \in R, \boldsymbol{x}_i \in R^K}{Max} \{u_i(\boldsymbol{x}_i) + x_i^0 | \boldsymbol{p} \cdot \boldsymbol{x}_i + x_i^0 = m_i\} = \underset{\boldsymbol{x}_i \in R^K}{Max}\, u_i(\boldsymbol{x}_i) - \boldsymbol{p} \cdot \boldsymbol{x}_i + m_i$$

b) Profit maximization: Each producer chooses $(\boldsymbol{y}_j^*, z_j^*)$ that maximizes the profit or the producer's surplus, $\pi_j(\boldsymbol{y}_j, z_j, \boldsymbol{p}^*)$, i.e.,

$$(\boldsymbol{y}_j^*, z_j^*) \equiv \underset{\boldsymbol{y}_j \in R^K, z_j \in Z}{argmax}\ \boldsymbol{p}^* \cdot \boldsymbol{y}_j - c_j(\boldsymbol{y}_j, z_j) = \underset{\boldsymbol{y}_j \in R^K, z_j \in Z}{argmax}\ \pi_j(\boldsymbol{y}_j, z_j, \boldsymbol{p}^*) \tag{12}$$

c) Market balance: the total demand equals the total supply for each commodity,

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} y_j^* \tag{13}$$

The Allocation Model

The objective function of the market allocation model is to maximize the net gains from trade measured by the social surplus, which equals the gross benefit minus the total cost. The market allocation $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$ is the solution to the following MIP problem that aims to maximize the social surplus, subject to the market balance condition.

$$V^* = \underset{\boldsymbol{x}_i \in R^K, \boldsymbol{y}_j \in R^K, z_j \in Z}{Max} \left\{ \sum_{i=1}^{I} u_i(\boldsymbol{x}_i) - \sum_{j=1}^{J} c_j(\boldsymbol{y}_j, z_j) \,\middle|\, \sum_{i=1}^{I} \boldsymbol{x}_i = \sum_{j=1}^{J} \boldsymbol{y}_j \right\} \tag{14}$$

Thus, the market allocation, $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$, is Pareto efficient.

The first fundamental theorem of welfare economics states that if the price vector $\boldsymbol{p}^*$ and the market allocation $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$ constitute a competitive equilibrium, then this allocation is Pareto efficient. That is, market equilibria are necessarily Pareto efficient. The second fundamental theorem of welfare economics is the converse of the first theorem stating the conditions under which for any Pareto efficient allocation, $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$, there exists a price vector $\boldsymbol{p}^*$ that supports the Pareto-efficient allocation at competitive equilibrium. The critical conditions turn out to be the convexity of preferences and production feasibility sets. That is, the utility functions and cost functions must be convex.

The Pricing Model

Quasi-equilibrium pricing is obtained from the solution to the convex relaxation of the allocation model. In the following, we consider three convex relaxation approaches: Lagrangian dualization, convex-hull relaxation and integer relaxation. First, the standard Lagrangian dualization formulation is as follows:

$$V^{LD} = \underset{\boldsymbol{p} \in R^K}{Inf} \left\{ \sum_{i=1}^{I} \widehat{\varphi}_i(\boldsymbol{p}) + \sum_{j=1}^{J} \hat{\pi}_j^{LD}(\boldsymbol{p}) \right\} \tag{15}$$

The main disadvantage with the classical approach lies in the computational difficulty of obtaining the maximum surplus function, $\hat{\pi}_j^{LD}(\boldsymbol{p})$, which involves solving a non-convex optimization problem for every given $\boldsymbol{p}$.

The second approach is convex-hull relaxation which is obtained by replacing each cost function $c_j(\boldsymbol{y}_j, z_j)$ by its convex envelope, denoted by $\check{c}_j(\boldsymbol{y}_j, z_j)$, and its constraint set by its convex hull $\boldsymbol{y}_j \in R^K, z_j \in \bar{Z} \equiv [0,1]$ where $\bar{Z}$ is the convex hull of Z.

$$V^{CH} = \underset{\boldsymbol{x}_i \in R^K, \boldsymbol{y}_j \in R^K, z_j \in \bar{Z}}{Max} \left\{ \sum_{i=1}^{I} u_i(\boldsymbol{x}_i) - \sum_{j=1}^{J} \check{c}_j(\boldsymbol{y}_j, z_j) \middle| \sum_{i=1}^{I} \boldsymbol{x}_i = \sum_{j=1}^{J} \boldsymbol{y}_j \right\} \tag{16}$$

Since the problem in (16) is now a convex optimization problem, its dual solution is well defined.

$$V^{CH} = \underset{\boldsymbol{p} \in R^K}{Inf} \left\{ \sum_{i=1}^{I} \widehat{\varphi}_i(\boldsymbol{p}) + \sum_{j=1}^{J} \hat{\pi}_j^{CH}(\boldsymbol{p}) \right\} \tag{17}$$

The computation of the convex envelope of the cost function remains a difficult challenge. The relaxation methods, based on convex-hull and Lagrangian dual, generally produce equivalent results, $V^{CH} = V^{LD}$. But, the main drawback with these two approaches is that they are computationally prohibitive.

The third approach, integer relaxation, is a fairly simple method that generally produces good approximations to Lagrangian dual and convex-hull relaxation. Integer relaxation entails the linearization of the integer variables in the MIP problem by converting binary integer variables in the original MIP problem into continuous real variables in $\bar{Z} = [0,1]$. The integer relaxation of the MIP model can be simply implemented by switching the feasibility set from Z to $\bar{Z}$ as follows:

$$V^{IR} = \underset{\boldsymbol{x}_i \in R^K, \boldsymbol{y}_j \in R^K, z_j \in \bar{Z}}{Max} \left\{ \sum_{i=1}^{I} u_i(\boldsymbol{x}_i) - \sum_{j=1}^{J} c_j(\boldsymbol{y}_j, z_j) \middle| \sum_{i=1}^{I} \boldsymbol{x}_i = \sum_{j=1}^{J} \boldsymbol{y}_j \right\} \tag{18}$$

Since the problem in (18) is a convex optimization problem, its dual problem is well defined:

$$V^{IR} = \underset{\boldsymbol{p} \in R^K}{Inf} \left\{ \sum_{i=1}^{I} \widehat{\varphi}_i(\boldsymbol{p}) + \sum_{j=1}^{J} \hat{\pi}_j^{IR}(\boldsymbol{p}) \right\} \tag{19}$$

Theorem 1. If the cost function $c_i(\boldsymbol{y}_j, z_j)$ is positive homogeneous of degree one in $(\boldsymbol{y}_j, z_j)$ for $j \in J$, then $V^{CH} = V^{LD} = V^{IR} \equiv V^{**}$.

Proof:

From Lemma 1, $\hat{\pi}_j^{LD}(\boldsymbol{p}) = \hat{\pi}_j^{CH}(\boldsymbol{p}) = \hat{\pi}_j^{IR}(\boldsymbol{p})$.

Substituting this result into (15), (17) and (19) yields $V^{LD} = V^{CH} = V^{IR} \equiv V^{**}$. QED

The difference $V^{**} - V^{*}$ equals the duality gap.

From the second fundamental theorem, under non-convex conditions, a price vector may not exist to support a Pareto-efficient allocation $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$ as a competitive equilibrium. However, the convexified allocation model is a well-defined, convex optimization problem that would produce as its solution a price vector $\boldsymbol{p}^{**}$ and market allocation vector $(\boldsymbol{x}^{**}, \boldsymbol{y}^{**}, \boldsymbol{z}^{**})$, forming a competitive equilibrium. Such a competitive equilibrium, based on the convexified allocation model, is called a quasi-equilibrium. By invoking the Shapley-Folkman Theorem, Starr (1969) shows that the discrepancy between the Pareto-efficient allocation, $(\boldsymbol{x}^{*}, \boldsymbol{y}^{*}, \boldsymbol{z}^{*})$ and the quasi-equilibrium allocation, $(\boldsymbol{x}^{**}, \boldsymbol{y}^{**}, \boldsymbol{z}^{**})$, is bounded from above by an amount that does not increase with the size of the market (e.g., the number of agents); hence the discrepancy, divided by the size of the market, approaches zero as the market size increases to infinity. Later in the paper, we extend this result using the duality gap as a measure of the discrepancy.

The Settlement Rule

The settlement rule aims to support the efficient allocation solution $(\boldsymbol{x}^{*}, \boldsymbol{y}^{*}, \boldsymbol{z}^{*})$ and the quasi-equilibrium pricing solution, $\boldsymbol{p}^{**}$**,** in such a way that at the price vector, $\boldsymbol{p}^{**}$, the competitive equilibrium conditions in (11) and (12) are satisfied – each consumer maximizes the consumer's surplus, and each producer maximizes the producer's surplus – while the efficient allocation $(\boldsymbol{x}^{*}, \boldsymbol{y}^{*}, \boldsymbol{z}^{*})$ meets the market balance condition (13). The solution support payments are defined as follows:

$$\begin{aligned} &\Delta\varphi_i(\boldsymbol{x}^{*}, \boldsymbol{p}^{**}) \equiv \hat{\varphi}_i(\boldsymbol{p}^{**}) - \varphi_i(\boldsymbol{x}_{\boldsymbol{i}}^{*}, \boldsymbol{p}^{**}) \\ &\Delta\pi_j\left(\boldsymbol{y}_{\boldsymbol{j}}^{*}, z_{\boldsymbol{j}}^{*}, \boldsymbol{p}^{**}\right) \equiv \hat{\pi}_j(\boldsymbol{p}^{**}) - \pi_j\left(\boldsymbol{y}_{\boldsymbol{j}}^{*}, z_{\boldsymbol{j}}^{*}, \boldsymbol{p}^{**}\right) \end{aligned} \tag{20}$$

The total solution support payment can be collected as taxes or fees from consumers and producers in proportion to the consumers' and producers' surplus in such a way that will not change their behavior. Alternatively, a two-part tariff with fixed subscription fees could produce minimum economic distortions. This is an example of the standard non-linear pricing problem. (Wilson 1992; Chao 2012).

For a more intuitive interpretation, the solution support payments in (20) can be rewritten explicitly in relation to the quasi-equilibrium allocation as follows:

$$\begin{aligned} &\Delta\varphi_i(\boldsymbol{x}^{*}, \boldsymbol{p}^{**}) = \varphi_i(\boldsymbol{x}_{\boldsymbol{i}}^{**}, \boldsymbol{p}^{**}) - \varphi_i(\boldsymbol{x}_{\boldsymbol{i}}^{*}, \boldsymbol{p}^{**}) \\ &\Delta\pi_j\left(\boldsymbol{y}_{\boldsymbol{j}}^{*}, z_{\boldsymbol{j}}^{*}, \boldsymbol{p}^{**}\right) = \pi_j\left(\boldsymbol{y}_{\boldsymbol{j}}^{**}, z_{\boldsymbol{j}}^{**}, \boldsymbol{p}^{**}\right) - \pi_j\left(\boldsymbol{y}_{\boldsymbol{j}}^{*}, z_{\boldsymbol{j}}^{*}, \boldsymbol{p}^{**}\right) \end{aligned} \tag{21}$$

Given the quasi-equilibrium price, p^{**}, the solution support payment settles the difference between: (1) the surplus that the consumers and producers get under and efficient allocation (x^{*}, y^{*}, z^{*}), and (2) the maximum surplus they can get if they are allowed to choose the allocation (x^{**}, y^{**}, z^{**}). Therefore, the total solution support payment is an appropriate measure of the discrepancy between the quasi-equilibrium allocation and the Pareto-efficient allocation.

Lemma 2. The total solution support payment equals the duality gap:

$$\sum_{i=1}^{I} \Delta\varphi_i(\boldsymbol{x}_i^{*}, \boldsymbol{p}^{**}) + \sum_{j=1}^{J} \Delta\pi_j\left(\boldsymbol{y}_{\boldsymbol{j}}^{*}, z_{\boldsymbol{j}}^{*}, \boldsymbol{p}^{**}\right) = V^{**} - V^{*} \tag{22}$$

Proof: Summing up (20) over i and j and using (14) and (15), we derive*:*

$$\sum_{i=1}^{I} \Delta\varphi_i(\boldsymbol{x}_i^{*}, \boldsymbol{p}^{**}) + \sum_{j=1}^{J} \Delta\pi_j\left(\boldsymbol{y}_{\boldsymbol{j}}^{*}, z_{\boldsymbol{j}}^{*}, \boldsymbol{p}^{**}\right) \tag{23}$$

$$= \left(\sum_{i=1}^{I} \hat{\varphi}_i(\boldsymbol{p}^{**}) + \sum_{j=1}^{J} \hat{\pi}_j(\boldsymbol{p}^{**})\right) - \left(\sum_{i=1}^{I} \varphi_i(\boldsymbol{x}_i^*, \boldsymbol{p}^{**}) + \sum_{j=1}^{J} \pi_j(\boldsymbol{y}_j^*, z_j^*, \boldsymbol{p}^{**})\right)$$

$$= V^{**} - \left(\sum_{i=1}^{I} u_i(\boldsymbol{x}_i^*) - \sum_{j=1}^{J} c_j(\boldsymbol{y}_j^*, z_j^*)\right) = V^{**} - V^*$$

QED.

Next, treating the duality gap as a measure of the discrepancy between the quasi-equilibrium allocation $(\boldsymbol{x}^{**}, \boldsymbol{y}^{**}, \boldsymbol{z}^{**})$ and the efficient allocation, $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$, we extend the previous results in Starr (1969) by showing that the duality gap, and thus the total solution support payment, is bounded from above. When divided by the total social surplus, the ratio will approach zero as the market size (measured by the number of consumers) increases to infinity.

Define:

$$\hat{\rho} \equiv \max_{\theta_j \in \Theta, \boldsymbol{y}_j \in R^K, z_j \in \bar{Z}} \{c(\boldsymbol{y}_j, z_j | \theta_j) - \check{c}(\boldsymbol{y}_j, z_j | \theta_j)\} \geq 0.$$

$$\varepsilon = \min_i [\hat{\varphi}_i(\boldsymbol{p}^{**})] > 0; \text{ and } \epsilon = \min_i [\hat{\pi}_j(\boldsymbol{p}^{**})] \geq 0. \tag{24}$$

Theorem 2. Given that $\hat{\rho} \geq 0$ and $\varepsilon + \epsilon > 0$, then A) $V^{**} - V^* \leq (K+2)\,\hat{\rho}$ and B) $\lim_{I \to \infty} (V^{**} - V^*)/V^{**} = 0$

Proof:

Invoking the Shapley-Folkman Theorem, Aubin and Ekeland (1976) show that if $I + J > K + 2$, the duality gap is bounded from above by $(K+2)\,\hat{\rho}$, which depends on the number of commodities but independent of the number of buyers and sellers, I and J.

$$0 \leq V^{**} - V^* \leq (K+2)\,\hat{\rho} \tag{25}$$

Then, we can write:

$$V^{**} = \sum_{i=1}^{I} \hat{\varphi}_i(\boldsymbol{p}^{**}) + \sum_{j=1}^{J} \hat{\pi}_j \;(\boldsymbol{p}^{**}) \geq I\varepsilon + J\epsilon > 0 \tag{26}$$

Dividing both sides of (25) by V^{**} and using (26), we obtain:

$$0 \leq \frac{V^{**} - V^{*}}{V^{**}} \leq \frac{(K+2)\,\hat{\rho}}{I\varepsilon + J\epsilon} \tag{27}$$

By taking the limit as $I \rightarrow \infty$, we prove the result (B). QED

The above result suggests that under quasi-equilibrium pricing, the revenue insufficiency problem is contained and will become relatively small as the market grows in size.

3. Incentive Compatibility

In this section, we examine the incentive properties for quasi-equilibrium pricing. For simplicity of exposition, we consider a single commodity, sealed-bid auction design with a uniform price. We assume that consumers are price takers and share an identical demand function denoted by $D(p)$. Using Roy's identity, we obtain:

$$\varphi'(p) = -D(p) \tag{28}$$

We assume that each firm has access to private information on technology type, denoted by θ_j, which is independently and identically distributed on Θ. Following a common assumption in the theory of incentive with asymmetric information, we assume that each firm's cost function is affine-linear: $c(y_j, z_j|\theta_j) = c(\theta_j)y_j + \phi(\theta_j)z_j = c_j y_j + \phi_j z_j$, where c_j and ϕ_j denote the marginal and fixed costs. Each firm's capacity is normalized to one unit, and the production possibility set is: $y_j \leq z_j \in \{0,1\}$. The optimal profit function is obtained:

$$\hat{\pi}_j\ (p) = \left(p - c_j - \phi_j\right)^{+} \tag{29}$$

From Hotelling's lemma, we write the supply function:

$$S_j(p) = \hat{\pi}_j'(p) = \begin{cases} 1, \text{if } p \geq c_j + \phi_j \\ 0, \text{if } p < c_j + \phi_j \end{cases} \tag{30}$$

The expected supply function is defined as:

$$\bar{S}(p) = E\{S_j(p)\} = Prob\{c(\theta_j) + \phi(\theta_j) \leq p\} \tag{31}$$

The auction rule comprises the bid format, the selection rule, the pricing rule and the settlement rule. At the start of the auction, after observing the private signal, θ_j, each firm submits a two-part bid $(\acute{c}_j, \acute{\phi}_j) \in \left\{\left(c(\acute{\theta}), \phi(\acute{\theta})\right) \middle| \acute{\theta} \in \acute{\Theta}\right\}$. A bidding strategy $(\acute{c}_j, \acute{\phi}_j) \equiv \left[c(\acute{\theta}_j|\theta_j), \phi(\acute{\theta}_j|\theta_j)\right]$ is a mapping from the signal space $\Theta \times \Theta$ to the feasible bidding set when the true type is θ_j and the submitted bid type is $\acute{\theta}_j$. The two-part bid format is consistent with the cost structure.

Following the tradition of Debreu and Scarf (1963), our analysis employs the tool of replica economy. The n-fold replica economy, denoted by $\langle \mathbb{M}(n) \rangle$, which is a sequence of markets with n consumers and n producers, so we have $I = J = n$. We assume that the consumers have the same demand function and the producers are identically and independently distributed on Θ.

The selection rule determines the allocation $(\acute{\boldsymbol{x}}^*, \acute{\boldsymbol{y}}^*, \acute{\boldsymbol{z}}^*)$ based on the market allocation model which maximizes the total social surplus.

$$V^* = \underset{x \in R, \boldsymbol{y}_j \in R, z_j \in Z}{Max} \left\{ n[u(x)] - \sum_{j=1}^{n} \left(\acute{c}_j y_j + \acute{\phi}_j z_j \right) \middle| nx = \sum_{j=1}^{n} y_j \, , y_j \le z_j, \forall \, j \right\} \tag{32}$$

The pricing rule determines $\acute{p}^{**}$ based on the pricing model:

$$\hat{V}^{**} \equiv \underset{\acute{\boldsymbol{p}} \in R^K}{min} \left\{ n\hat{\varphi}(\acute{p}) + \sum_{j=1}^{n} \hat{\pi}_j(\acute{p}) \right\} \tag{33}$$

At the close of the auction, the buyers and sellers are contracted according to the bid-based market allocation, $(\acute{\boldsymbol{x}}^*, \acute{\boldsymbol{y}}^*, \acute{\boldsymbol{z}}^*)$. The settlement will be based on the bid-based market price vector, $\acute{p}^{**}$, plus the solution support payments. As a result, each buyer gains the consumer's surplus $\hat{\varphi}_i(\acute{p}^{**})$, and each seller gains the producer's surplus $\hat{\pi}_j(\acute{p}^{**})$.

Let $\hat{\pi}\left(\acute{p}^{**} | \acute{\theta}_j, \theta_j\right)$ denote the actual producer's surplus when the producer has a true type θ_j and bid its type as $\acute{\theta}_j$ and $\acute{p}^{**}\left(\acute{\boldsymbol{\theta}}\right)$ denote the commodity price. Let $\acute{p}_{-j}^{**}$ be a solution to the following pricing problem:

$$\hat{V}_{-j}^{**} \equiv \underset{\acute{\boldsymbol{p}} \in R^K}{min} \left\{ n\hat{\varphi}(\acute{\boldsymbol{p}}) + \sum_{\substack{\xi=1 \\ \xi \neq j}}^{n} \hat{\pi}_j(\acute{\boldsymbol{p}}) \right\} \tag{34}$$

In an auction market with a unit demand, there exists a price solution $\acute{p}^{**}$ that satisfies

$$\hat{V}^{**} - \hat{V}_{-j}^{**} = \hat{\pi}_j(\acute{\boldsymbol{p}}^{**}) \tag{35}$$

The famous example for a uniform pricing rule that satisfies (35) is the single-unit Vickrey auction in which the lowest bid is selected as the winner, and the second lowest bid sets the price. A more general rule is the Vickrey-Clark-Grove mechanism that compensates each producer, *j*, its incremental contribution to the total value, $\hat{V}^{**} - \hat{V}_{-j}^{**}$, though the VCG mechanism, in general, does not produce a uniform price. The VCG mechanism is DSIC, which is defined as follows:

Definition: A market mechanism is *dominant strategy incentive compatible (DSIC* if $\hat{\pi}_j\left(\acute{\boldsymbol{p}}^{**}\left(\theta_j, \acute{\boldsymbol{\theta}}_{-j}\right) | \theta_j, \theta_j\right) \geq \hat{\pi}_j\left(\acute{\boldsymbol{p}}^{**}\left(\acute{\theta}_j, \acute{\boldsymbol{\theta}}_{-j}\right) | \acute{\theta}_j, \theta_j\right)$ for any $\left(\acute{\theta}_j, \acute{\boldsymbol{\theta}}_{-j}\right) \in \Theta^n$.

Intuitively, DSIC means that for every producer, truthful bidding is the optimal strategy, independent of the strategies of the other producers.

On the surface, the result of uniform pricing under Vickrey auction seems to contradict the proposition that with a finite number of agents in an exchange economy, no market mechanism based on uniform pricing can assure incentives for competitive price-taking behavior (Hurwicz, 1972). Actually, the Vickrey pricing rule is due to the assumption on price-insensitive fixed demand, which violates the non-satiation assumption in the standard general equilibrium model. Indeed, when the demand function is price sensitive, DSIC cannot be assured for market mechanisms under uniform pricing. For example, if the consumer preference is strictly convex and producers are diverse, then, $\acute{p}^{**} \neq \acute{p}_{-j}^{**}$, and (35) cannot hold. Nonetheless, Postlewaite and Roberts (1976) show that in large economies, incentive compatibility for price-taking behavior would prevail when the number of consumers increases to infinity. In the following, we extend these insights to replica markets with non-convexities.

Next, in a sequence of replica markets, we show that the quasi-equilibrium pricing mechanism would approach the VCG mechanism in the limit and become DSIC when the market size increases to infinity.

Theorem 4. For the sequence of replica markets $\langle \mathbb{M}(n) \rangle$, the quasi-equilibrium pricing mechanism converges to a VCG mechanism, $\lim_{n\to\infty} \hat{V}^{**} - \hat{V}_{-j}^{**} - \hat{\pi}_j(\acute{\boldsymbol{p}}^{**}) = 0$, and thus is DSIC in the limit.

Proof:

First, from the first-order condition of (15), we obtain:

$$n\hat{\varphi}'(\acute{p}^{**}) + \sum_{j=1}^{n} \hat{\pi}'_j(\acute{p}^{**}) = 0 \tag{36}$$

From (28) and (30), we obtain:

$$-D(\acute{p}^{**}) + \frac{1}{n}\sum_{j=1}^{n} S_j(\acute{p}^{**}) = 0 \tag{37}$$

From the strong law of large numbers, we obtain:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} S_j(\acute{p}^{**}) = \bar{S}_j(\acute{p}^{**}) \;\; a.s. \, (\text{almost surely}) \tag{38}$$

Substituting (38) into (37) and taking the limit as $n \to \infty$, we have:

$$D(\acute{p}^{**}) = \bar{S}(\acute{p}^{**}) \tag{39}$$

Let p^* be the solution to the last equation in (39) is independent on n:

$$\lim_{n\to\infty} \acute{p}^{**} = p^* \; a.s. \tag{40}$$

For each $\boldsymbol{\theta}_{-\boldsymbol{j}} \in \Theta^{n-1}$, from similar steps as (37) – (40), we obtain:

$$\lim_{\boldsymbol{n}\to\infty} \acute{p}_{-j}^{**} = p^* \; a.s. \tag{41}$$

From (40) and (41), we obtain:

$$\lim_{\boldsymbol{n}\to\infty} \acute{p}^{**} = \lim_{\boldsymbol{n}\to\infty} \acute{p}_{-j}^{**} = p^* \; a.s. \tag{42}$$

From (33), (34) and (42), we have:

$$\lim_{n\to\infty} \hat{V}^{**} - \hat{V}_{-j}^{**} - \hat{\pi}_j(\boldsymbol{\acute{p}}^{**})$$

$$= \lim_{n\to\infty} \left[n\hat{\varphi}(\boldsymbol{\acute{p}}^{**}) + \sum_{\xi=1}^{n} \hat{\pi}_{\xi}\ (\boldsymbol{\acute{p}}^{**}) \right] - \left[n\hat{\varphi}(\boldsymbol{\acute{p}}_{-i}^{**}) + \sum_{\substack{\xi=1 \\ \xi\neq j}}^{n} \hat{\pi}_{\xi}\ \left(\boldsymbol{\acute{p}}_{-\xi}^{**}\right) \right] - \hat{\pi}_j(\boldsymbol{\acute{p}}^{**}) \tag{43}$$

$$= \lim_{n\to\infty} \left[n\hat{\varphi}(p^{*}) + \sum_{\xi=1}^{n} \hat{\pi}_{\xi}\ (p^{*}) \right] - \left[n\hat{\varphi}(p^{*}) + \sum_{\substack{\xi=1 \\ \xi\neq j}}^{n} \hat{\pi}_{\xi}\ (p^{*}) \right] - \hat{\pi}_j\ \ (p^{*}) = 0\ a.s.$$

QED

4. Wholesale Electricity Markets

In this section, we illustrate the above results in a pool-based wholesale electricity market. The allocation model is represented by the dispatch model that produces efficient commitment and dispatch solutions within a transmission network, with the objective of maximizing social welfare measured by the sum of consumers' and producers' surplus or total surplus, also called market surplus. The pricing model is represented by a convex relaxation of the allocation model. The settlement rule determines the solution support payments, known as the uplift payments.

The electricity market framework features an electric transmission network and daily markets with multiple hourly periods. For simplicity, we will consider the standard unit commitment and economic dispatch problem with a DC flow model without losses or ramping-rate constraints and no operating reserve requirements. For convenience, consumers and generators are indexed by node as representative agents at each node in the grid with no loss of generality. Including operating reserves and joint determination of energy and reserve prices raises no fundamental issues. Similarly, application in real time would require dynamic optimization rolling the solution forward, which can be accommodated but would complicate the notation.[15]

Notation

Index:

$i, j \in N \equiv \{1, \dots, n\}$: *index for nodes or generating units in the transmission network*

$l \in L \equiv \{1, \dots, L\}$: *index for lines in the transmission network*

$t \in T \equiv \{1, \dots, T\}$: *index for time*

Decision variables:

[15] The real-time market clearing model includes multiple periods and look ahead. Prices are calculated for a time window containing the day of the real-time market on a rolling basis as day progresses.

$\boldsymbol{x}_t = (x_{1t}, \ldots, x_{nt})$: *the demand levels of customers* $i \in N$ *at time* t[16]

$\boldsymbol{y}_t = (y_{1t}, \ldots, y_{nt})$: *the generation levels of units* $i \in N$ *at time* t

$\boldsymbol{z}_t = (z_{1t}, \ldots, z_{nt})$:: $w_{it} \in \{0,1\}$ *is the on-off state for unit* $i \in N$ *at time* t

$\boldsymbol{u}_t = (u_{1t}, \ldots, u_{nt})$:: $u_{it} \in \{0,1\}$s the start-up decision for unit $i \in N$ at time t

$\boldsymbol{p}_t = (p_{1t}, \ldots, p_{nt})$:: the price at node $i \in N$ and time t

Parameters:

$\boldsymbol{\lambda_t} = (\lambda_{0t}, \lambda_{1t} \ldots, \lambda_{Lt})$: *the shadow price vector for the reference node* (λ_{0t}) *and power lines* $(\lambda_{1t} \ldots, \lambda_{Lt})$.

$\boldsymbol{b} = (b_l)$: *the energy transmission capabilities for* $l \in L$

$\boldsymbol{K} = [\beta_{li}]$: *the power transfer distribution factors (d-fax) for* $i \in N$ *and* $l \in L$

G_{it}^M, G_{it}^m: *the maximum and minimum levels of economic generation of unit* $i \in N$ *at time* t

$B_i(x_{it})$: *the gross benefit function for the consumer at node* $i \in N$

$C_i(y_{it}, z_{it})$: *the variable cost function for unit* i *where* y_{it} *is the generation output and* z_{it} *is the on-off state of unit* $i \in N$

c_i^{SU}: *the start-up cost for unit* $i \in N$

c_i^{NL}: the no-load cost for unit $i \in N$

The Dispatch Model

The standard optimal unit commitment and economic dispatch model for market clearing is formulated as an MIP problem below:

$$v^* = \underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{u}}{Max} \sum_{t=1}^{T} \sum_{i=1}^{n} \left[B_i(x_{it}) - C_i(y_{it}, z_{it}) - c_{it}^{SU} u_{it} - c_{it}^{NL} z_{it}\right] \tag{44}$$

subject to (for $i = 1, \ldots, n$ and $t = 1, \ldots, T$):

$$\sum_{i=1}^{n} (x_{it} - y_{it}) = 0 \qquad \perp \lambda_{0t} \tag{45}$$

[16] The demand is defined as net of behind-the-meter generation or self-scheduled generation. Self-scheduling reveals the participant's willingness to behave as a pure price taker. Self-scheduled units are not included in the economic dispatch choices, and therefore are not treated as dispatch choices that set prices in the pricing model.

$$\sum_{i=1}^{n} \beta_{\ell i}(x_{it} - y_{it}) \le b_\ell \qquad \perp \lambda_{\ell t} \tag{46}$$

$$z_{it} G_i^m \le y_{it} \le z_{it} G_i^M \tag{47}$$

$$z_{it} - z_{i,t-1} \le u_{it} \tag{48}$$

$$x_{it}, y_{it} \in R^+ \text{ and } u_{it}, z_{it} \in \mathrm{Z} \equiv \{0,1\} \tag{49}$$

The objective function in (44) is to maximize the social surplus, or the gross consumer benefit minus the total cost, which equals the sum of the variable cost, start-up cost and no-load cost. Equation (45) represents the demand and supply balancing condition. Constraints (46) represent the transmission network power flow balancing condition, where $\boldsymbol{K}$ denotes the power flow distribution factors based on the physical Kirchhoff laws, and $\boldsymbol{b}$ denotes the power flow capacity of transmission lines. Constraints (47) represent the minimum and maximum generation capacity limits. Constraints (48) represent the state transition condition for changing unit on-off status. Constraints (49) represent the feasibility set for unit commitment and dispatch variables.

The unit commitment and economic dispatch instructions are based on the optimal solution of the dispatch model denoted by $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*, \boldsymbol{u}^*)$. In the presence of non-convexity, however, there exists no single set of prices that could support the optimal dispatch solution alone. In (47) and (48), the commitment variables, z_{it}, do not usually appear in a standard unit commitment and economic dispatch model. These variables are included as redundant indicator variables that do not change the feasibility set in the dispatch model, but they would form a convex cone with the integer relaxation creating a dual model needed for efficient pricing.

The Pricing Model

The nodal price is the sum of the price at the reference node and the transmission congestion revenue from the constrained power lines:

$$\boldsymbol{p_{it}} = \lambda_{0t} + \sum_{\ell=1}^{L} \beta_{\ell i} \boldsymbol{\lambda}_{\ell t} \tag{50}$$

We extend the definition of the consumer's surplus, the producer's surplus for $i \in N$ and the transmission congestion revenue $\mu(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p})$ as follows:

$$\varphi_i \ (\boldsymbol{x}, \boldsymbol{p_i}) \equiv \sum_{t=1}^{T} [B_i(x_{it}) - p_{it} x_{it}] \tag{51}$$

$$\pi_i \ (\boldsymbol{y_i}, \boldsymbol{z_i}, \boldsymbol{u_i}, \boldsymbol{p_i}) \equiv \sum_{t=1}^{T} p_{it} y_{it} - C_i(y_{it}, z_{it}) - c_{it}^{SU} u_{it} - c_{it}^{NL} z_{it} \tag{52}$$

$$\mu(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}) \equiv \sum_{t=1}^{T} \boldsymbol{p_t} \cdot (\boldsymbol{x_t} - \boldsymbol{y_t}) = \sum_{t=1}^{T} \sum_{\ell=1}^{L} b_\ell \boldsymbol{\lambda}_{\ell t} \tag{53}$$

$$\breve{\pi}_i\ (\boldsymbol{y_i}, \boldsymbol{z_i}, \boldsymbol{u_i}, \boldsymbol{p_i}\ ) \equiv \sum_{t=1}^{T} p_{it}\, y_{it}\ - \breve{C}_i(y_{it}, z_{it}) - c_{it}^{SU} u_{it} - c_{it}^{NL} z_{it} \tag{54}$$

The equality in (53) follows the nodal-flow congestion revenue equivalence theorem.[17] We assume that transmission-right holders are price takers. The maximum consumer's surplus and producer's surplus are defined as conjugate functions as follows:

$$\hat{\varphi}_i\ (\boldsymbol{p_i}\ ) \equiv \max_{x_i \in R^T} \sum_{t=1}^{T} [B_i(x_{it}\ ) - p_{it}\, x_{it}\ ] \tag{55}$$

$$\pi_i^*(\boldsymbol{z_i}, \boldsymbol{u_i}, \boldsymbol{p_i}\ ) \equiv \underset{y_i \in R^T}{max} \{\pi_i\ (\boldsymbol{y_i}, \boldsymbol{z_i}, \boldsymbol{u_i}, \boldsymbol{p_i}\ ) | (47)\} \tag{56}$$

$$\hat{\pi}_i^{CH}(\boldsymbol{p}) \equiv \max_{y_i \in R^T, \boldsymbol{z_i}, \boldsymbol{u_i} \in \bar{Z}^T} \{\breve{\pi}_i\ (\boldsymbol{y_i}, \boldsymbol{z_i}, \boldsymbol{u_i}, \boldsymbol{p_i}\ ) | (47) - (48)\} \tag{57}$$

$$\hat{\pi}_i^{LD}(\boldsymbol{p_i}\ ) \equiv \max_{\boldsymbol{z_i}, \boldsymbol{u_i} \in Z^T} \{\pi_i^*(\boldsymbol{z_i}, \boldsymbol{u_i}, \boldsymbol{p_i}\ ) \big| (48)\} \tag{58}$$

$$\hat{\pi}_i^{IR}(\boldsymbol{p_i}\ ) \equiv \underset{\boldsymbol{z_i}, \boldsymbol{u_i} \in \bar{Z}^T}{max} \{\pi_i^*(\boldsymbol{z_i}, \boldsymbol{u_i}, \boldsymbol{p_i}\ ) \big| (48)\} \tag{59}$$

$$\hat{\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}) \equiv \sum_{t=1}^{T} \boldsymbol{p_t} \cdot (\boldsymbol{x_t} - \boldsymbol{y_t}) = \sum_{t=1}^{T} \sum_{\ell=1}^{L} b_\ell \boldsymbol{\lambda}_{\ell t} \tag{60}$$

Note that the main difference between the dispatch model and the pricing model is that the integer constraints on commitment variables in the dispatch model are relaxed in the pricing model in such a way that commitment variables stay within the continuous unit interval between zero and one. The pricing solution, or the energy price vector, $\boldsymbol{p}_t^{**}$, for $t = 1, \ldots T$, is supplemented by the uplift payments to be described later, so that no one would have an incentive or can gain financially by deviating from the efficient commitment and dispatch solution, $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*, \boldsymbol{u}^*)$.

The expressions of Lagrangian dualization, convex-hull relaxation and integer relaxation are as follows,

$$v^{LD} = \underset{\boldsymbol{\lambda}, \boldsymbol{p}}{Inf} \left\{ \sum_{i=1}^{n} \hat{\varphi}_i\ (\boldsymbol{p_i}\ ) + \hat{\pi}_i^{LD}(\boldsymbol{p_i}\ ) + \hat{\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}) | (50) \right\} \tag{61}$$

$$v^{CH} = \underset{\boldsymbol{\lambda}, \boldsymbol{p}}{Inf} \left\{ \sum_{i=1}^{n} \hat{\varphi}_i\ (\boldsymbol{p_i}\ ) + \hat{\pi}_i^{CH}(\boldsymbol{p_i}\ ) + \hat{\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}) | (50) \right\} \tag{62}$$

$$v^{IR} = \underset{\boldsymbol{\lambda}, \boldsymbol{p}}{Inf} \left\{ \sum_{i=1}^{n} \hat{\varphi}_i\ (\boldsymbol{p_i}\ ) + \hat{\pi}_i^{IR}(\boldsymbol{p_i}\ ) + \hat{\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}) | (50) \right\} \tag{63}$$

[17] Chao and Peck (1996)

The Settlement Rule

The solution support or the uplift payment is calculated as the difference between the maximum surplus and the actual surplus for each generator, consumer and transmission provider:

$$\begin{aligned} &\Delta\varphi_i\ \ (\boldsymbol{x}_i^*, \boldsymbol{p}_i^{**}) \equiv \hat{\varphi}_i\ \ (\boldsymbol{p}_i^{**}) - \varphi_i\ \ (\boldsymbol{x}_i^*, \boldsymbol{p}_i^{**}) \\ &\Delta\pi_i\ \ (\boldsymbol{y}_i^*, \boldsymbol{z}_i^*, \boldsymbol{u}_i^*, \boldsymbol{p}_i^{**}) \equiv \hat{\pi}_i\ \ (\boldsymbol{p}_i^{**}) - \pi_i\ \ (\boldsymbol{y}_i^*, \boldsymbol{z}_i^*, \boldsymbol{u}_i^*, \boldsymbol{p}_i^{**}) \\ &\Delta\mu(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{p}^{**}) \equiv \hat{\mu}(\boldsymbol{x}^{**}, \boldsymbol{y}^{**}, \boldsymbol{p}^{**}) - \hat{\mu}(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{p}^{**}) \end{aligned} \tag{64}$$

The following assumption extends the positive homogeneity property to the constraints in (47) – (48).

Property A: For $i = 1, \ldots, n$, we assume that:

1) $C_i(y_i, z_i)$ is a positive homogeneous function of degree one in (y_i, z_i) and convex in y_i.

2) The constraint set (47) is a convex cone separable in i and t.

3) The constraint set (48) is structurally a unimodal network-flow model.

Note that Property A1 does not require the cost function $C_i(y_i, z_i)$ to be convex in (y_i, z_i), for that assumption would imply that the integer relaxation is equivalent to the convex-hull relaxation. In most applications, this property could be secured through formulation techniques. For example, given cost function $C_i(y_i)$, we could construct a function that is positive homogeneous of degree one $\widehat{C}_i(y_{i,}, z_i) \equiv z_i C_i(y_i/z_i)$, so that $\widehat{C}_i(y_{i,}, z_i) = C_i(y_i)$, when $z_i = 1$. In general, $\widehat{C}_i(y_{i,}, z_i)$ is not necessarily a convex function in (y_i, z_i).

In the electricity market, the variable generation cost function is commonly formulated as a linear programming problem as follows:

$$C_i^0(y_i) \equiv \min_g \left\{ \sum_{\ell=1}^L c_{i\ell} g_{i\ell} \,\middle|\, \sum_{\ell=1}^L g_{i\ell} = y_i, 0 \le g_{i\ell} \le \delta_{i\ell} \right\} \text{ for } y_i \in \left[0, G_i^M\right] \tag{65}$$

where $\delta_{i\ell}$ is the MW step size of the incremental capacity, and $\sum_{\ell=1}^L \delta_{i\ell} = G_i^M$ is the unit's generation capacity.

To include the commitment variable z_i in the variable cost function, one approach is to modify all incremental capacity constraints:

$$C_i^1(y_i, z_i) \equiv \min_x \left\{ \sum_{\ell=1}^L c_{i\ell} g_{i\ell} \,\middle|\, \sum_{\ell=1}^L g_{i\ell} = y_i, 0 \le g_{i\ell} \le z_i \delta_{i\ell} \right\} \text{ for } y_i \in \left[0, z_i G_i^M\right] \tag{66}$$

The cost function in (66) is positive homogeneous of degree one in $(y_{i,}, z_i)$, and $C_i^1(y_i, z_i) = C_i^0(y_i)$ if $z_i = 1$. The homogeneity property follows directly from the standard result that the optimal value of a linear program is positive homogeneous of degree one with respect to the right-hand side. In this specific example, $C_i^1(y_i, z_i)$ turns out to be a convex function in (y_i, z_i), and thus the integer relaxation is equivalent to the convex-hull relaxation.

Property A2 and A3 can be extended to incorporate most of the constraints in electricity markets such as generation capacity, state transition, and even the time-coupling minimum up/down are formulated to be positive homogeneous of degree one. However, it remains an open research question whether ramp-rate constraints and potentially some other constraints may continue to be a barrier for integer relaxation to obtain an exact convex-hull pricing solution. Nonetheless, the following results support the observation that integer relaxation generally provides a close, and sometimes even an exact, approximation to convex-hull pricing.

Lemma 2. If Property A is satisfied, then $\hat{\pi}_i^{CH}(\boldsymbol{p}) = \hat{\pi}_i^{LD}(\boldsymbol{p}) = \hat{\pi}_i^{IR}(\boldsymbol{p}) \equiv \hat{\pi}_i\ (\boldsymbol{p})$.

Proof:

First, note that $\pi_i^*(\boldsymbol{z_i}, \boldsymbol{u_i}, \boldsymbol{p_i}\ )$ is a linear function in $\boldsymbol{u_i}$, and both the function and the constraint set (47) is separable in i and t. Since a homogenous function over a convex cone remains a homogenous function, the linearity of $\pi_i^*(\boldsymbol{z_i}, \boldsymbol{u_i}, \boldsymbol{p_i}\ )$ follows from Lemma 1. The second part of the proof follows from the integrality theorem for network flow (Ford and Fulkerson, 1962) in linear programming, since the problems in (57) – (59) are structurally equivalent to a maximal-flow problem on a unimodal, network-flow model with integer-valued capacity, hence there exists an integer-valued optimal solution. This means that the integrality constraints are non-binding, and thus can be relaxed without affecting the optimal objective value. QED

Theorem 5. If Property A is true, then $v^{CH} = v^{LD} = v^{IR} \equiv v^{**}$.

Proof:

The result follows directly from Lemma 2 by applying its results of (61) – (63). QED

5. Discussion

Convex-hull pricing represents a theoretical ideal that supports efficient dispatch while minimizing the total uplift payment (Gribik, Hogan, and Pope, 2007). However, this approach is computationally challenging under the best of circumstances, and it would be even more challenging to apply it in the short time frame (5-minute) required for the real-time wholesale electricity markets. The computational challenge has motivated several recent research efforts. Chen and Wang (2018) investigate fast methods to estimate the convex envelope of the cost functions (including start-up and no-load costs) in piece-wise linear models. Hua and Baldick (2018) identify conditions where the cost function is piece-wise linear or quadratic, and the ramping constraints are not binding, which allows for faster computation of the convex envelope of the cost function, as well as the pricing model. In contrast, the integer relaxation method does not rely on the estimate of the convex envelope of the cost function. Instead, it is based on a simple modification of the MIP formulation under the conditions of Property A, in such that the pricing model is essentially the same as the dispatch model, except the complicating commitment constraints are relaxed while producing close, and sometimes exact, approximations to the convex-hull pricing solution. As suggested by Kuang, Lamadrid and Zuluaga (2019) and others, dynamic ramping constraints/costs remain a barrier to implement full, convex-hull pricing and warrant future research.

The current pricing method underlying the wholesale electricity market design in the RTOs and ISOs is based on the dual solution of the restricted linear programming problem obtained by fixing the commitment decisions at the optimal levels in the MIP problem as follows:

$$v^0 = \underset{\boldsymbol{\lambda},\boldsymbol{p}}{Inf}\left\{\sum_{i=1}^{n} \hat{\varphi}_i\ (\boldsymbol{p_i}\ ) + \pi_i^*(\boldsymbol{z}_i^*, \boldsymbol{u}_i^*, \boldsymbol{p_i}\ ) + \hat{\mu}(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{p}) | (50)\right\} \tag{67}$$

This implementation employs a single run of the security-constrained economic dispatch (SCED) MIP model for both dispatch and pricing purposes. In the SCED run, only units that satisfy convex conditions are eligible to set price, and the costs of non-convex units (such as lumpy, block-loaded units) associated with commitment decisions are not reflected in the restricted LMP model. Thus, the prices obtained from the restricted dual formulation of (67) ignore the commitment costs of resources needed to serve demand, essentially treating them as sunk costs. The restricted formulation does not support incentive compatibility in the dispatch and pricing solution. Even with make-whole uplift payments, there exist units that are profitable to run but may be instructed to go offline, but these units have an incentive to self-schedule or submit bids with inflexible operating parameters to evade operator instructions. In essence, the usual argument for the standard LMP approach avoids this discussion by assuming that there are no non-convexities that affect the total cost of the dispatch, and that the LMP prices are all that is needed to support the dispatch. When this assumption is not true, the uplift payments that are required (to avoid creating incentives for market participants to deviate from the economic dispatch) cannot prevent them from deviating from economic commitment through self-schedule.

In April 2019, FERC authorized the use of integer relaxation as a just and reasonable pricing method for fast-start units in PJM's energy markets,[18] and units that are required to stay off-line according to the optimal SCED solution due to non-convexity conditions will be paid their lost opportunity costs.

Example

In this example, we examine a two-bus network with two generating units over a congested transmission line. We assume that the transmission capacity between the two nodes is 120 MW. Table 1 provides the numerical assumptions for the two generating units in this example.

Table 1. Numerical Assumptions for Example 1

	Node 1	Node 2
Load (MW)	50	150
Minimum Generation (MW)	0	50
Maximum Generation (MW)	250	50
Start-up cost ($)	0	100
Variable cost ($/MWh)	20	40

In Table 1, the unit at Node 1 is more efficient with zero fixed cost and a lower variable cost than the unit at Node 2, and its capacity (250 MW) is sufficient to meet the system load at 200 MW. Transmission capacity is apparently valuable. If the transmission capacity were unlimited, the output from the efficient unit will serve the entire market demand without congestion. In that event, the LMPs would be set uniformly at the two nodes and equal $20/MWh. However, given the transmission capacity of 120 MW, the optimal system dispatch would have to call on the less efficient and less flexible unit at Node 2.

[18] PJM (2019a, 2019b)

Table 2 shows the market clearing and pricing results. The nodal price difference reflects a positive value of the transmission line, though a uplift is needed to cover the underutilized transmission capacity caused by the lumpiness of the unit at Node 2: $(\$42-\$20)\times(120-100)=\$440$. The market clearing results achieve the minimum total uplift payment.

Table 2. Market Clearing Results under Integer Relaxation

	Node 1	Node 2
Demand (MW)	50	150
Supply (MW)	150	50
Price ($/MWh)	20	42
Uplift payment ($)	0	0
Transmission Uplift ($)	440	

6. Conclusion

Non-convexity has long been recognized a fundamental challenge for competitive markets to attain Pareto-efficient allocation. In electricity markets, the current LMP-based wholesale electricity market is vulnerable to distorted incentives under non-convex conditions. Through separated allocation and pricing models supported, quasi-equilibrium pricing mechanisms extend efficient pricing principles from a convex to a non-convex market environment in ways that ensure economic efficiency, incentive compatibility and revenue sufficiency. This paper contributes two new insights. First, we show that quasi-equilibrium pricing is dominant strategy incentive compatible in the limit and the total side payment, divided by the total surplus, and approaches zero when the market size (measured by the number of consumers) increases to infinity. Second, we identify conditions under which the integer relaxation method can produce close, or even exact, approximations to convex-hull pricing. In April 2019, FERC authorized the use of integer relaxation as a just and reasonable pricing method for fast-start units in PJM's energy markets.

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